On Some New Seminormed Sequence Spaces Defined by a Sequence of Orlicz Functions

Çiğdem A. Bektaş and Gülcan Atıci

Department of Mathematics, Firat University, 23119 Elazığ, Turkey

Reprint requests to Ç. A. B. or G. A.; E-mail:cigdemas78@hotmail.com or gatici23@hotmail.com

Z. Naturforsch. **65a,** 919 – 923 (2010); received July 24, 2009 / revised November 25, 2009

In this paper, we define the new generalized difference sequence spaces $c_0(\Delta_v^m, M, u, p, q)$, $c(\Delta_v^m, \mathrm{M}, u, p, q)$, and $\ell_\infty(\Delta_v^m, M, u, p, q)$. We also study some inclusion relations between these spaces.

Key words: Difference Sequence Spaces; Orlicz Function; Seminorm.
AMS 2000: 40A05, 40C05, 46A45

1. Introduction

From a mathematical point of view, transition from classical mechanics to quantum mechanics amounts to, among other things, passing from the commutative algebra of classical observables to the non-commutative algebra of quantum mechanical observables. Recall that in classical mechanics an observable (e. g. energy, position, momentum, etc.) is a function on a manifold called the phase space of the system. A little more than 50 years after these developments, Alain Connes realized that a similar procedure can in fact be applied to areas of mathematics where the classical notions of space (e. g. measure space, locally compact space, or a smooth space) loses its applicability and pertinence and can be replaced by a new idea of space, represented by a non-commutative algebra. Conne's theory, which is generally known as non-commutative geometry, is a rapidly growing new area of mathematics that interacts with and contributes to many disciplines in mathematics and physics. For a recent survey, see Conne's article [1].

Examples of such interactions and contributions include: theory of operator algebras, index theory of elliptic operators, algebraic and differential topology, number theory, standard model of elementary particles, quantum Hall effect, renormalization in quantum field theory, and string theory.

As cited above operator algebras are presently one of the dynamic areas of mathematics.

In this paper, by using difference operator, we define the new generalized difference sequence spaces. Nowadays, operator algebra and operator theory play an important role in different areas of mathematics, and its applications, particularly in mathematics, physics, and numerical analysis.

Hopefully, this study about operator theory serves for researchers who carry research in various fields of science.

Let w denote the set of all sequences $x = (x_k)$ and ℓ_∞, c, and c_0 be the linear spaces of bounded, convergent, and null sequences with real terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$.

Throughout the article $w(X)$, $\ell_\infty(X)$, $c(X)$, and $c_0(X)$ will represent the spaces of all bounded, convergent, and null X valued sequence spaces. For $X = C$, the field of complex numbers, these represent the corresponding scalar valued sequence spaces.

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \le \lambda M(x)$ for all λ with $0 \le \lambda \le 1$.

Lindenstrauss and Tzafriri [2] used the idea of the Orlicz function to define what is called an Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

which is a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

Several authors, including Alsaedi and Bataineh [3], Bektaş [4, 5] and many others, have studied sequence spaces defined by the Orlicz function.

The difference sequence spaces were introduced by Kızmaz [6] and the concept was generalized by Et and Çolak [7]. After then Et and Esi [8] extended the difference sequence spaces to the sequence spaces

$$X(\Delta_v^m) = \{x = (x_k) : (\Delta_v^m x_k) \in X\}$$

for $X = \ell_\infty$, c, or c_0, where $v = (v_k)$ be any fixed sequence of non-zero complex numbers and

$$\left(\Delta_v^m x_k\right) = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1}) ,$$

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} v_{k+i} x_{k+i} \text{ for all } k \in N.$$

The sequence spaces $\Delta_v^m(\ell_\infty)$, $\Delta_v^m(c)$, and $\Delta_v^m(c_0)$ are Banach spaces normed by

$$\|x\|_\Delta = \sum_{i=1}^{m} |v_i x_i| + \|\Delta_v^m x\|_\infty .$$

Definition 1.1. Let q_1, q_2 be seminorms on a vector space X. Then q_1 is said to be stronger than q_2 if whenever (x_n) is a sequence such that $q_1(x_n) \to 0$, than also $q_2(x_n) \to 0$. If each is stronger than the others, then q_1 and q_2 are said to be equivalent [9].

Lemma 1.2. Let q_1 and q_2 be seminorms on a linear space X. Then q_1 is stronger than q_2 if and only if there exists a constant K such that $q_2(x) \le K q_1(x)$ for all $x \in X$ [9].

Definition 1.3. A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \le 1$, [10].

Definition 1.4. A sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces, [10].

Remark 1.5. From the two above definitions it is clear that 'A sequence space E is solid implies that E is monotone', [11].

Let $u = (u_k)$ be a sequence of non-zero scalars. Then for a sequence space E, the multiplier sequence space $E(u)$, associated with the multiplier sequence u is defined as

$$E(u) = \{(x_k) \in w : (u_k x_k) \in E\} .$$

The studies on the multiplier sequence spaces are done by Çolak [12], Bektaş et al. [13], Et [14] and many others.

Let U be the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ for all $k \in N$.

Let $\mathrm{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be any sequence of strictly positive real numbers, X be a seminormed space with the seminorm q, and $u \in U$. Then we define

$$c_0(\Delta_v^m, \mathrm{M}, u, p, q) = \left\{x \in w(X) : \lim_{k\to\infty} u_k \left[M_k\left(q\left(\frac{\Delta_v^m x_k}{\rho}\right)\right)\right]^{p_k} = 0 \text{ for some } \rho > 0\right\},$$

$$c(\Delta_v^m, \mathrm{M}, u, p, q) = \left\{x \in w(X) : \lim_{k\to\infty} u_k \left[M_k\left(q\left(\frac{\Delta_v^m x_k - \ell}{\rho}\right)\right)\right]^{p_k} = 0, \text{ for some } \rho > 0,\ \ell \in X\right\},$$

$$\ell_\infty(\Delta_v^m, \mathrm{M}, u, p, q) = \left\{x \in w(X) : \sup_k u_k \left[M_k\left(q\left(\frac{\Delta_v^m x_k}{\rho}\right)\right)\right]^{p_k} < \infty, \text{ for some } \rho > 0\right\}.$$

If we take $q(x) = |x|$, then we have the sequence spaces

$$c_0(\Delta_v^m, \mathrm{M}, u, p) = \left\{x \in w : \lim_{k\to\infty} u_k \left[M_k\left(\frac{|\Delta_v^m x_k|}{\rho}\right)\right]^{p_k} = 0, \text{ for some } \rho > 0\right\},$$

$$c(\Delta_v^m, \mathrm{M}, u, p) = \left\{x \in w : \lim_{k\to\infty} u_k \left[M_k\left(\frac{|\Delta_v^m x_k - \ell|}{\rho}\right)\right]^{p_k} = 0, \text{ for some } \rho > 0,\ \ell \in C\right\},$$

$$\ell_\infty(\Delta_v^m, \mathrm{M}, u, p) = \left\{x \in w : \sup_k u_k \left[M_k\left(\frac{|\Delta_v^m x_k|}{\rho}\right)\right]^{p_k} < \infty, \text{ for some } \rho > 0\right\}.$$

If $u_k = 1$, $M_k(x) = x$ for all $k \in N$ and $q(x) = |x|$, then the sequence spaces $c_0(\Delta_v^m, \mathrm{M}, u, p, q)$, $c(\Delta_v^m, \mathrm{M}, u, p, q)$, and $\ell_\infty(\Delta_v^m, \mathrm{M}, u, p, q)$ reduce to the sequence spaces $c_0(p)(\Delta_v^m)$, $c(p)(\Delta_v^m)$, and $\ell_\infty(p)(\Delta_v^m)$ which were defined and studied by Et et al. [15].

If $u_k = 1$, $M_k = M$ for all $k \in N$, then the sequence spaces $c_0(\Delta_v^m, \mathrm{M}, u, p, q)$, $c(\Delta_v^m, \mathrm{M}, u, p, q)$, and $\ell_\infty(\Delta_v^m, \mathrm{M}, u, p, q)$ reduce to the sequence spaces

$c_0(\Delta_v^m, M, p, q)$, $c(\Delta_v^m, M, p, q)$, and $\ell_\infty(\Delta_v^m, M, p, q)$, respectively.

If we take $u_k = 1$ for all $k \in N$, we shall write $c_0(\Delta_v^m, \mathrm{M}, p, q)$, $c(\Delta_v^m, \mathrm{M}, p, q)$, and $\ell_\infty(\Delta_v^m, \mathrm{M}, p, q)$ instead of $c_0(\Delta_v^m, \mathrm{M}, u, p, q)$, $c(\Delta_v^m, \mathrm{M}, u, p, q)$, and $\ell_\infty(\Delta_v^m, \mathrm{M}, u, p, q)$, respectively.

2. Main Results

In this section we introduce some new results.

Theorem 2.1. Let $p = (p_k)$ be a bounded sequence, then the spaces $c_0(\Delta_v^m, \mathrm{M}, u, p, q)$, $c(\Delta_v^m, \mathrm{M}, u, p, q)$, and $\ell_\infty(\Delta_v^m, \mathrm{M}, u, p, q)$ are linear spaces over the field of complex numbers $\mathbb{C}$.

Proof is trivial.

Theorem 2.2. The spaces $c_0(\Delta_v^m, \mathrm{M}, u, p, q)$, $c(\Delta_v^m, \mathrm{M}, u, p, q)$, and $\ell_\infty(\Delta_v^m, \mathrm{M}, u, p, q)$ are paranormed spaces with

$$g(x) = \inf\left\{\rho^{p_n/H} > 0 : \left\{\sup_k \left[u_k M_k\left(q\left(\frac{\Delta_v^m x_k}{\rho}\right)\right)\right]^{p_k}\right\}^{1/H} \leq 1\right\},$$

where $H = \max(1, \sup_k p_k)$.

The proof is routine verification by using standard arguments and therefore omitted.

Theorem 2.3. $\ell_\infty(\Delta_v^m, \mathrm{M}, p)$ is a complete paranormed space with

$$g(x) = \inf\left\{\rho^{p_n/H} > 0 : \left\{\sup_k \left[M_k\left(\frac{|\Delta_v^m x_k|}{\rho}\right)\right]^{p_k}\right\}^{1/H} \leq 1\right\}.$$

Proof. Let (x^i) be any Cauchy sequence in $\ell_\infty(\Delta_v^m, \mathrm{M}, p)$. Let $r, x_0 > 0$ be fixed. Then for each $\frac{\varepsilon}{rx_0} > 0$, there exists a positive integer N such that

$$g(x^i - x^j) < \frac{\varepsilon}{rx_0}$$

for all $i, j \geq N$. Using the definition of paranorm, we get

$$\left\{\sup_k \left[M_k\left(\frac{\left|\Delta_v^m x_k^i - \Delta_v^m x_k^j\right|}{g(x^i - x^j)}\right)\right]^{p_k}\right\}^{1/H} \leq 1$$

for all $i, j \geq N$. Thus

$$\sup_k \left[M_k\left(\frac{\left|\Delta_v^m x_k^i - \Delta_v^m x_k^j\right|}{g(x^i - x^j)}\right)\right]^{p_k} \leq 1$$

for all $i, j \geq N$. It follows that

$$M_k\left(\frac{\left|\Delta_v^m x_k^i - \Delta_v^m x_k^j\right|}{g(x^i - x^j)}\right) \leq 1$$

for each $k \geq 0$ and for all $i, j \geq N$. For $r > 0$ with $M_k(\frac{rx_0}{2}) \geq 1$, we have

$$M_k\left(\frac{\left|\Delta_v^m x_k^i - \Delta_v^m x_k^j\right|}{g(x^i - x^j)}\right) \leq M_k\left(\frac{rx_0}{2}\right).$$

This implies that

$$\left|\Delta_v^m x_k^i - \Delta_v^m x_k^j\right| \leq \frac{rx_0}{2} \cdot \frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2}.$$

Hence, $(\Delta_v^m x_k^i)$ is a Cauchy sequence in $\mathbb{R}$. Therefore, for each $\varepsilon (0 < \varepsilon < 1)$, there exists a positive integer N such that $\left|\Delta_v^m x_k^i - \Delta_v^m x_k^j\right| < \varepsilon$, for all $i \geq N$.

Using the continuity of M_k for each k, we can find that

$$\left\{\sup_{k \geq N} \left[M_k\left(\frac{\left|\Delta_v^m x_k^i - \lim_{j\to\infty} \Delta_v^m x_k^j\right|}{\rho}\right)\right]^{p_k}\right\}^{1/H} \leq 1.$$

Thus

$$\left\{\sup_{k \geq N} \left[M_k\left(\frac{\left|\Delta_v^m x_k^i - \Delta_v^m x_k\right|}{\rho}\right)\right]^{p_k}\right\}^{1/H} \leq 1.$$

Taking infimum of such ρ's we get

$$\inf\left\{\rho^{p_n/H} : \left\{\sup_{k \geq N} \left[M_k\left(\frac{\left|\Delta_v^m x_k^i - \Delta_v^m x_k\right|}{\rho}\right)\right]^{p_k}\right\}^{1/H} \leq 1\right\} < \varepsilon$$

for all $i \geq N$ and $j \to \infty$. Since $(x^i) \in \ell_\infty(\Delta_v^m, \mathrm{M}, p)$ and M_k is an Orlicz function for each k and therefore continuous, we get that $x \in \ell_\infty(\Delta_v^m, \mathrm{M}, p)$.

This completes the proof of the theorem.

Theorem 2.4. Assume that $0 < p_k \le r_k < \infty$, for each k. Then we have $c_0(\Delta_v^m, \mathrm{M}, p, q) \subseteq c_0(\Delta_v^m, \mathrm{M}, r, q)$, $c(\Delta_v^m, \mathrm{M}, p, q) \subseteq c(\Delta_v^m, \mathrm{M}, r, q)$, and $\ell_\infty(\Delta_v^m, \mathrm{M}, p, q) \subseteq \ell_\infty(\Delta_v^m, \mathrm{M}, r, q)$.

Proof. We prove it for $c_0(\Delta_v^m, \mathrm{M}, p, q) \subseteq c_0(\Delta_v^m, \mathrm{M}, r, q)$ and the other cases will follow on applying similar techniques. Let $x \in c_0(\Delta_v^m, \mathrm{M}, p, q)$. Then there exists some $\rho > 0$ such that

$$\lim_{k\to\infty}\left[M_k\left(q\left(\frac{\Delta_v^m x_k}{\rho}\right)\right)\right]^{p_k} = 0.$$

This implies that $M_k\left(q\left(\frac{\Delta_v^m x_k}{\rho}\right)\right) \le 1$ for sufficiently large k, since all M_k are non-decreasing. Hence, we get

$$\lim_{k\to\infty}\left[M_k\left(q\left(\frac{\Delta_v^m x_k}{\rho}\right)\right)\right]^{r_k} \le \lim_{k\to\infty}\left[M_k\left(q\left(\frac{\Delta_v^m x_k}{\rho}\right)\right)\right]^{p_k} = 0,$$

i. e., $x \in c_0(\Delta_v^m, \mathrm{M}, r, q)$.

Corollary 2.5.

(i) Let $0 < \inf p_k \le p_k \le 1$. Then $c_0(\Delta_v^m, \mathrm{M}, p, q) \subseteq c_0(\Delta_v^m, \mathrm{M}, q)$.

(ii) Let $1 \le p_k \le \mathrm{supp_k} < \infty$. Then $c_0(\Delta_v^m, \mathrm{M}, q) \subseteq c_0(\Delta_v^m, \mathrm{M}, p, q)$.

Theorem 2.6. Let M be an Orlicz function, then

(i) $c_0(\Delta_v^m, \mathrm{M}, p, q) \subset \ell_\infty(\Delta_v^m, \mathrm{M}, p, q)$,

(ii) $c(\Delta_v^m, \mathrm{M}, p, q) \subset \ell_\infty(\Delta_v^m, \mathrm{M}, p, q)$.

Proof. The inclusion $c_0(\Delta_v^m, \mathrm{M}, p, q) \subset \ell_\infty(\Delta_v^m, \mathrm{M}, p, q)$ is obvious. The second inclusion follows from the following inequality. Let $x \in c(\Delta_v^m, \mathrm{M}, p, q)$.

$$\left[M\left(q\left(\frac{\Delta_v^m x_k}{\rho}\right)\right)\right]^{p_k} \le \frac{D}{2^{p_k}}\left[M\left(q\left(\frac{\Delta_v^m x_k - \ell}{\rho_1}\right)\right)\right]^{p_k} + \frac{D}{2^{p_k}}\left[M\left(q\left(\frac{\ell}{\rho_1}\right)\right)\right]^{p_k} \le D\left[M\left(q\left(\frac{\Delta_v^m x_k - \ell}{\rho_1}\right)\right)\right]^{p_k} + D\max\left\{1, \sup\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\ell}{\rho_1}\right)\right)\right]^{\mathrm{G}}\right\},$$

where $\rho = 2\rho_1$, $\mathrm{sup_k\, p_k = G}$ and $D = \max\{1, 2^{G-1}\}$.

The proof of the following results are easy thus omitted.

Theorem 2.7. Let $M = (M_k)$ and $\mathrm{T} = (T_k)$ be any two sequence of Orlicz functions. Then we have $Z(\Delta_v^m, \mathrm{M}, u, p, q) \cap Z(\Delta_v^m, \mathrm{T}, u, p, q) \subset Z(\Delta_v^m, \mathrm{M} + \mathrm{T}, u, p, q)$, for $Z = c_0$ or c or ℓ_∞.

Theorem 2.8. Let $\mathrm{M} = (M_k)$ be a sequence of Orlicz functions. For any two seminorms q_1 and q_2, we have $Z(\Delta_v^m, \mathrm{M}, u, p, q_1) \cap Z(\Delta_v^m, \mathrm{M}, u, p, q_2) \subset Z(\Delta_v^m, \mathrm{M}, u, p, q_1 + q_2)$, for $Z = c_0$ or c or ℓ_∞.

Theorem 2.9. Let $\mathrm{M} = (M_k)$ be a sequence of Orlicz functions and any two seminorms q_1 and q_2. If q_1 is stronger than q_2, then $Z(\Delta_v^m, \mathrm{M}, u, p, q_1) \subset Z(\Delta_v^m, \mathrm{M}, u, p, q_2)$, for $Z = c_0$ or c or ℓ_∞.

Theorem 2.10. Let X stands for c_0 or c or ℓ_∞. Then $X(\Delta_v^{m-1}, \mathrm{M}, q) \subset X(\Delta_v^m, \mathrm{M}, q)$ and inclusions is strict. In general $X(\Delta_v^i, \mathrm{M}, q) \subset X(\Delta_v^m, \mathrm{M}, q)$ for all $i = 1, 2, \ldots, m-1$ and the inclusions are strict.

Proof. We give the proof for $X = \ell_\infty$ only and the other cases can be proved by using similar arguments. Let $x \in \ell_\infty(\Delta_v^{m-1}, \mathrm{M}, q)$. Then we have

$$\sup_k\left[M_k\left(q\left(\frac{\Delta_v^{m-1} x_k}{\rho}\right)\right)\right] < \infty.$$

Since M_k are non-decreasing and convex functions, q is a seminorm and Δ_v^m is linear, we have

$$\left[M_k\left(q\left(\frac{\Delta_v^m x_k}{\rho}\right)\right)\right] = \left[M_k\left(q\left(\frac{\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1}}{\rho}\right)\right)\right] \le \frac{1}{2}\left[M_k\left(q\left(\frac{\Delta_v^{m-1} x_k}{\rho_1}\right)\right)\right] + \frac{1}{2}\left[M_k\left(q\left(\frac{\Delta_v^{m-1} x_{k+1}}{\rho_1}\right)\right)\right] < \infty, \text{ where } \rho = 2\rho_1.$$

Thus $\ell_\infty(\Delta_v^{m-1}, \mathrm{M}, q) \subset \ell_\infty(\Delta_v^m, \mathrm{M}, q)$. Proceeding in this way one will have $\ell_\infty(\Delta_v^i, \mathrm{M}, q) \subset \ell_\infty(\Delta_v^m, \mathrm{M}, q)$ for all $i = 1, 2, \ldots, m-1$. The sequence $x = (k^m)$, for example, belongs to $\ell_\infty(\Delta_v^m, \mathrm{M}, q)$, but does not belong to $\ell_\infty(\Delta_v^{m-1}, \mathrm{M}, q)$ for $M_k(x) = x$, $v_k = 1$ for all $k \in N$ and $q(x) = |x|$. Therefore the inclusions are strict.

Theorem 2.11.

(i) The spaces $c_0(M, u, p, q)$ and $\ell_\infty(M, u, p, q)$ are solid and hence are monotone.

(ii) The space $c(M, u, p, q)$ is not monotone and therefore neither solid nor perfect.

Proof. We give the proof for $c_0(M, u, p, q)$. Let $x \in c_0(M, u, p, q)$ and (α_k) be sequences of scalars such that $|\alpha_k| \leq 1$ for all $k \in N$. Then we have

$$u_k\left[M_k\left(q\left(\frac{\alpha_k x_k}{\rho}\right)\right)\right]^{p_k} \leq u_k\left[M_k\left(q\left(\frac{x_k}{\rho}\right)\right)\right]^{p_k} \to 0$$

as $k \to \infty$. Hence, $\alpha x \in c_0(M,u,p,q)$ for all sequence of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in N$, whenever $x \in c_0(M,u,p,q)$. The spaces are monotone follows from the Remark 1.5.

Acknowledgement

The authors wish to express their thanks to Professor Mikail Et for his suggestions and guidance in the preparation of this paper.

[1] A. Connes, in: Noncommutative Geometry Year 2000. Highlights of Mathematical Physics, London 2000, p. 49; Amer. Math. Soc. Providence, RI 2002.
[2] J. Lindenstrauss and L. Tzafriri, Israel J. Math. **10**, 379 (1971).
[3] R. S. Alsaedi and A. H. A. Bataineh, Int. Math. Forum **2**, 167 (2007).
[4] Ç. A. Bektaş, Southeast Asian Bull. Math. **28**, 195 (2004).
[5] Ç. A. Bektaş, Math. Slovaca (in press).
[6] H. Kızmaz, Canad. Math. Bull. **24**, 169 (1981).
[7] M. Et and R. Çolak, Soochow J. Math. **21**, 377 (1995).
[8] M. Et and A. Esi, Bull. Malaysian Math. Sc. Soc. **23**, 25 (2000).
[9] A. Wilansky, Functional Analysis. Blaisdell Publishing Company, New York 1964.
[10] P. K. Kamthan and M. Gupta. Sequence spaces and series. Marcel Dekker Inc., New York 1981.
[11] J. Boos, Classical and Modern Methods in Summability. Oxford Univ. Press, Oxford 2000.
[12] R. Çolak, Comm. Fac. Sci. Univ. Ankara Ser. A1. Math. Statist. **38**, 35 (1989).
[13] Ç. A. Bektaş, M. Et, and R. Çolak, J. Math. Anal. Appl. **292**, 423 (2004).
[14] M. Et, Taiwanese J. Math. **10**, 865 (2006).
[15] M. Et, H. Altınok, and Y. Altın, Appl. Math. Comput. **154**, 167 (2004).